B.Sc. Part I Paper II

Differential calculus

Successive Differentiation continued

Q If $y = x^2 e^x$, prove that

$$\frac{d^n y}{dx^n} = \frac{1}{2} n(n-1)\frac{d^2 y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)y.$$

Solution

$\because y = x^2 e^x$. Diff. w.r.t. x, we get

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2 e^x) = x^2 e^x + 2x e^x$$
$$= e^x(x^2 + 2x) \quad \text{——— (1)}$$

$$\therefore \frac{d^2y}{dx^2} = e^x(x^2+2x) + e^x(2x+2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x(x^2 + 4x + 2) \quad \text{——— (2)}$$

Again, $y = x^2 e^x$

Diff n times (by Leibnitz's theorem) with respect to x, we have

$$\frac{d^n y}{dx^n} = [e^x x^2]_n = [e^x]_n x^2 + {}^nC_1 [e^x]_{n-1} [x^2]_1 + {}^nC_2 [e^x]_{n-2} [x^2]_2 + \cdots$$

$$\therefore \frac{d^n y}{dx^n} = e^x \cdot x^2 + n e^x \cdot 2x + \frac{n(n-1)}{2} \cdot e^x \cdot 2 + 0$$

[Students :- note that other terms are 0 because $[x^2]_3$, $[x^2$ ... are equal to 0]

$$\Rightarrow \frac{d^n y}{dx^n} = e^x \left[x^2 + 2nx + n(n-1)\right] \quad \text{---(3)}$$

Now,

$$RHS = \frac{1}{2} n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-$$

$$= \frac{1}{2} n(n-1) e^x (x^2+4x+2) - n(n-2) e^x (x^2+2x) + \frac{1}{2}(n-1)(n-2) x^2 e^x$$

$$= \frac{1}{2} e^x \left[n(n-1)(x^2+4x+2) - 2n(n-2)(x^2+2x) + (n-1)(n-2)x^2\right]$$

$$= \frac{1}{2} e^x \left[n^2(x^2+4x+2) - n(x^2+4x+2) - 2n^2(x^2+2x) + 4n(x^2+2x) + (n^2-3n+2)x^2\right]$$

$$= \frac{e^x}{2}\left[2n^2 - 2n + 4nx + 2x^2\right]$$

$$= e^x \left[x^2 + 2nx + n^2 - n\right]$$

$$\Rightarrow RHS = e^x\left[x^2 + 2nx + n(n-1)\right] = \frac{d^n y}{dx^n} \text{ [using (3)]}$$

$$= LHS \text{ Proved.}$$

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Q If $y = a\cos(\log x) + b\sin(\log x)$,

Prove that

$$x^2 y_2 + x y_1 + y = 0 \quad \text{and}$$

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$$

Soln :

$$y = a\cos(\log x) + b\sin(\log x) \quad \text{---(1)}$$

Differentiating it with respect to x, we get

$$y_1 = -a\sin(\log x) \times \frac{1}{x} + b\cos(\log x) \times \frac{1}{x}$$

$$\Rightarrow x y_1 = b\cos(\log x) - a\sin(\log x) \quad \text{---(2)}$$

Differentiating with respect to x, we get

$$\Rightarrow x y_2 + y_1 = -b\sin(\log x) \times \frac{1}{x} - a\cos(\log x) \times \frac{1}{x}$$

$$\Rightarrow x^2 y_2 + x y_1 = -[b\sin(\log x) + a\cos(\log x)]$$

$$= -y \quad [\text{using (1)}]$$

$$\therefore \boxed{x^2 y_2 + x y_1 + y = 0} \quad \text{Proved.}$$

Differentiating it n times with respect to x, we get

$$[y_2 x^2]_n + [y_1 x]_n + y_n = 0$$

$$\Rightarrow [y_2]_n x^2 + {}^nC_1 [y_2]_{n-1} [x^2]_1 + {}^nC_2 [y_2]_{n-2} [x^2]_2$$

$$+ \cdots + [y_1]_n x + {}^nC_1 [y_1]_{n-1} [x]_1 + \cdots$$

$$+ [y]_n = 0$$

[Students note that $[x^2]_3, [x^2]_4, \cdots = 0$

and $[x]_2, [x]_3, \cdots = 0$]

$$\Rightarrow y_{n+2} x^2 + n \times y_{n+1} \times 2x + \frac{n(n-1)}{\cancel{2}} \times y_n \times \cancel{2}$$

$$+ x y_{n+1} + n y_n + y_n = 0$$

$$\Rightarrow x^2 y_{n+2} + y_{n+1} [2nx + x] + y_n [n(n-1) + n + 1] = 0$$

$$\Rightarrow x^2 y_{n+2} + x(2n+1) y_{n+1} + (n^2+1) y_n = 0$$